Differential Geometry III: Riemannian Geometry

G. Moschidis

13.1 Let (S, g) be a 2-dimensional Riemannian manifold and let $p \in S$. For any $0 < r < \iota(p)$, we will denote with L_r the length of the "circle" of points of distance equal to r from p. Show that the sectional cruvature at p satisfies

$$K(p) = \lim_{r \to 0} \frac{3}{\pi} \frac{2\pi r - L_r}{r^3},$$

(note that $2\pi r$ is simply the length of a Euclidean circle of radius r). *(Hint: You might want to use Exercise 10.1 about the expression of g in polar coordinates around p.*

13.2 Let (S, g) be a 2-dimensional connected and complete Riemannian manifold and let $p \in S$.

(a) Let $\gamma : [0, +\infty) \to S$ be a unit-speed geodesic such that $\gamma(0) = p$ and let $\hat{n} \in \Gamma_\gamma$ be a unit vectorfield along γ which is orthogonal at every point to $\dot{\gamma}$. Show that $\hat{n}$ is parallel transported along γ. *(Hint: Compute the projections of $\nabla_{\dot{\gamma}} \hat{n}$ on $\dot{\gamma}$ and $\hat{n}$. Note that here one has to make use of the fact that the dimension of $T_{\gamma(t)} S$ is 2.)*

(b) With γ and $\hat{n}$ as above, let $J \in \Gamma_\gamma$ be a *Jacobi* vector field such that $J(0) = 0$ and $J \perp \dot{\gamma}$. Show that $J(t) = f(t)\hat{n}(t)$ for some $f : [0, +\infty) \to \mathbb{R}$ satisfying

$$\begin{cases} f'' + Kf = 0, \\ f(0) = 0, \quad f'(0) = \langle \nabla_{\dot{\gamma}} J, \hat{n} \rangle|_{t=0}, \end{cases}$$

where $K(t) = K|_{\gamma(t)}$ is the sectional curvature. Show that the ratio

$$\lambda(t) \doteq \frac{f'(t)}{f(t)}$$

(defined at all point where $J(t) \neq 0$) satisfies the Ricatti type equation

$$\lambda' = -K - \lambda^2.$$

(c) Assume that the curvature of (S, g) satisfies everywhere the lower bound

$$K \geqslant 1. \tag{1}$$

Show that along any geodesic γ as above, p has a conjugate point $\gamma(t_*)$ for some $t_* \leqslant \pi$. *(Hint: Show that $\lambda(t)$ satisfies the differential inequality $\frac{d}{dt} Arctan(\lambda(t)) \leqslant -1$ with initial condition $t\lambda(t) \xrightarrow{t \to 0^+} 1$.)*

(d*) Assume that on a Riemannian surface (S, g) as above (satisfying, in particular, (1)), there exists a point p with injectivity radius satisfying $\iota(p) \geqslant \pi$. Show that (S, g) is isometric to $(\mathbb{S}^2, g)$. *(Hint: Show first that the curvature K has to be everywhere equal to 1.)*

13.3 Let $(\mathcal{M}, g)$ be a Riemannian manifold. For any smooth vector field $X \in \Gamma(\mathcal{M})$, we will define the divergence $\text{div}X$ of X to be the function defined as the contraction

$$\text{div}X \doteq \text{tr}\nabla X = (\nabla_i X)^i.$$

Similarly, for any $\omega \in \Gamma^*(\mathcal{M})$, we will define

$$\text{div}\omega \doteq \text{div}(\omega^\sharp).$$

Note that, in local coordinates,

$$\text{div}\omega = g^{ij}(\nabla_i \omega)_j.$$

In this exercise, we will generalise the well-known divergence identity from multivariable calculus to the Riemannian setting. To this end, let ω be a smooth 1-form on $\mathcal{M}$ and $\Omega \subset \mathcal{M}$ be an open domain with compact closure and with a (possibly empty) piecewise smooth boundary $\partial\Omega$. Let also $\hat{n}$ be the unit normal to $\partial\Omega$ pointing out of Ω and $\bar{g}$ be the induced metric on $\partial\Omega$ (both defined at every point where $\partial\Omega$ is smooth). Our aim is to show that

$$\int_\Omega \text{div}\omega \ \text{dVol}_g = \int_{\partial\Omega} \omega(\hat{n}) \ \text{dVol}_{\bar{g}} \tag{2}$$

(a) Show that, in any local coordinate system, $\text{div}\omega$ takes the form

$$\text{div}\omega = \frac{1}{\sqrt{\det g}}\partial_i\big(g^{ij}\sqrt{\det g}\,\omega_j\big).$$

Hint: You might want to use the following formmula from linear algebra regarding the derivative of the determinant of a matrix-valued function: If $A = A(s)$ is a family of matrices, then

$$\frac{d}{ds}\det A = \det A \cdot tr\Big(A^{-1}\frac{dA}{ds}\Big).$$

(b) Assume that Ω has the property that $\text{clos}(\Omega)$ is covered by a cooridnate chart $(x^1, \dots, x^n)$, with respect to which it is of the form

$$\Omega = \bigcap_{i=1}^{n}\big\{a_-^i < x^i < a_+^i\big\}$$

for some constants $a_-^i < a_+^i$, $i = 1, \dots, n$ (i.e. Ω is a coordinate box). Note that, in this case,

$$\partial\Omega = \bigcup_{i=1}^{1} S_+^{(i)} \cup S_-^{(i)}, \quad \text{where } S_\pm^{(i)} = \{x^i = a_\pm^i\} \cap \{a_-^j \leqslant x^j \leqslant a_+^j,\ j \neq i\}.$$

Show that, in this case, (2) holds. *(Hint: Compute first $\hat{n}$ and $\bar{g}$ on $S_\pm^{(i)}$. Recall that $d\text{Vol}_g = \sqrt{\det g}\,\text{d}x^1 \dots \text{d}x^n$.)*

(c) Show that (2) holds for a general domain Ω with compact closure and with piecewise smooth boundary $\partial\Omega$. You might use as given the fact that any such domain Ω can be written as a finite union of coordinate boxes as in the previous step, i.e. there exists a finite number of open sets $\Omega_\alpha \subset \Omega$ satisfying the following properties:

1. $\cup_\alpha \mathrm{clos}(\Omega_\alpha) = \mathrm{clos}(\Omega)$,
2. $\Omega_\alpha \cap \Omega_\beta = \emptyset$ if $\alpha \neq \beta$,
3. For each Ω_α, there exists a coordinate system around Ω_α with respect to which Ω_α satisfies the assumptions of the previous step.

13.4 (The Bochner technique). Let $(\mathcal{M}, g)$ be a Riemannian manifold. For any smooth function $f : \mathcal{M} \to \mathbb{R}$, we will define the *Laplacian* $\Delta_g f$ of f by the formula

$$\Delta_g f \doteq \mathrm{div}(df).$$

(a) Show that $\Delta_g f = g^{ij}\mathrm{Hess}[f]_{ij}$.

(b) Show that if $\mathcal{M}$ is compact, then

$$\int_{\mathcal{M}} \Delta_g f \ \mathrm{dVol}_g = 0.$$

Hint: Use Exercise 13.3.

(c*) Let X be a vector field on $\mathcal{M}$. Show that

$$\Delta_g \langle X, X\rangle = 2\|\nabla X\|^2 - 2X(\mathrm{div}X) + 2\langle \mathrm{div}_1(\nabla X)_{ant}, X\rangle - 2\mathrm{Ric}(X,X), \qquad (3)$$

where

$$\big((\nabla X)_{ant}\big)_i^{\ j} = (\nabla_i X)^j + g^{jl}g_{im}(\nabla_l X)^m,$$

$$\big(\mathrm{div}_1(\nabla X)_{ant}\big)^j = g^{kl}\big(\nabla_k(\nabla X)_{ant}\big)_l^{\ j}$$

and

$$\|\nabla X\|^2 = g^{ij}g_{kl}\nabla_i X^k \nabla_j X^l.$$

Hint: You can start from the relation $\big(d\langle X, X\rangle\big)_i = 2\langle \nabla_i X, X\rangle$ *and calculate its divergence.*

(d*) Let X be a Killing vector field on $(\mathcal{M}, g)$ (recall that this implies that $\nabla_i X_j + \nabla_j X_i = 0$). Show that $(\nabla X)_{ant} = 0$ and $\mathrm{div}X = 0$. Show that if $\mathcal{M}$ is connected and compact and the Ricci curvature satisfies for any $p \in \mathcal{M}$ and any $V \in T_p\mathcal{M} \setminus 0$

$$\mathrm{Ric}(V,V) < 0,$$

then X has to be identically 0 *(Hint: Integrate* (3) *over* $\mathcal{M}$*.)*. Deduce that the group of isometries of a compact Riemannian manifold with negative Ricci curvature is discrete.